

PSEUDO-RIEMANNIAN SUBMANIFOLDS WITH 3-PLANAR GEODESICS

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Abstract

In the present paper we study pseudo-Riemannian submanifolds which have 3-planar geodesic normal sections. We consider W -curves (helices) on pseudo-Riemannian submanifolds. Finally, we give necessary and sufficient condition for a normal section to be a W -curve on pseudo-Riemannian submanifolds.

1 Introduction

¹ In a Riemannian manifold, a regular curve is called a helix if its first and second curvatures is constant and the third curvature is zero. In 1980 Ikawa investigated the condition that every helix in a Riemannian submanifold is a helix in the ambient space [11]. In a pseudo-Riemannian manifold, helices are defined by almost the same way as the Riemannian case. The same author also characterized the helices in Lorentzian submanifold [12].

An isometric immersion $f : M_r^n \rightarrow \mathbb{R}_s^N$ is said to be planar geodesic if the image of each geodesic of M_r lies in a 2-plane of \mathbb{R}_s^N . In the Riemannian case such immersions were studied and classified by Hong [10], Little [17], Sakamoto [22], Ferus [8] and others. Further, Blomstrom classified planar geodesic immersions with indefinite metric [5]. It has been shown that all parallel, planar geodesic surfaces in \mathbb{R}_s^N are the pseudo-Riemannian spheres, the Veronese surfaces and certain flat quadratic surfaces. Recently Kim studied minimal surfaces of pseudo-Euclidean spaces with geodesic normal sections. He proved that complete connected minimal surfaces in a 5-dimensional pseudo-Euclidean space with geodesic normal sections are totally geodesics or flat quadrics [13].

In the present work, we give some results toward a characterization of 3-planar geodesic immersions $f : M_r \rightarrow \mathbb{N}_s$ from an n -dimensional, connected pseudo-Riemannian manifold M_r into m -dimensional pseudo-Riemannian manifold \mathbb{N}_s . Further, We consider W -curves (helices) on pseudo-Riemannian submanifolds. Finally, we give necessary and sufficient condition for a normal section to be a W -curve on pseudo-Riemannian submanifolds.

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2 Basic Concepts

Let $f : M_r \rightarrow \mathbb{N}_s$ be an isometric immersion from an n -dimensional, connected pseudo-Riemannian manifold M_r of index r ($0 \leq r \leq n$) into m -dimensional pseudo-Riemannian manifold \mathbb{N}_s of index s . Let ∇ and $\tilde{\nabla}$ denote the covariant derivatives of M_r and \mathbb{N}_s respectively. Thus $\tilde{\nabla}_X$ is just the directional derivative in the direction X in \mathbb{N}_s . Then for tangent vector fields X, Y the *second fundamental form* h of the immersion f is defined by

$$h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y. \quad (1.1)$$

For a vector field ξ normal to M_r we put

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \quad (1.2)$$

where A_ξ is the shape operator of M_r and D is the normal connection of M_r . We have the following relation

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle. \quad (1.3)$$

The covariant derivatives of h denoted respectively by $\bar{\nabla}h$ and $\bar{\nabla} \bar{\nabla}h$ to be;

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \quad (1.4)$$

and

$$\begin{aligned} (\bar{\nabla}_W \bar{\nabla}_X h)(Y, Z) &= D_W((\bar{\nabla}_X h)(Y, Z)) - (\bar{\nabla}_{\nabla_W X} h)(Y, Z) - \\ &\quad - (\bar{\nabla}_X h)(\nabla_W Y, Z) - (\bar{\nabla}_X h)(Y, \nabla_W Z), \end{aligned} \quad (1.5)$$

where X, Y, Z and W are tangent vector fields over M_r and $\bar{\nabla}$ is the Vander Waerden-Bortolotti connection [6]. Then we obtain the Codazzi equation

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) = (\bar{\nabla}_Z h)(X, Y). \quad (1.6)$$

It is a well-known property that $\bar{\nabla}h$ is a trilinear symmetric form on M_r with values in the normal bundle $N(M_r)$ and it is called the *third fundamental form*. If $\bar{\nabla}h = 0$, then the second fundamental form is said to be *parallel* [9] (*i.e.* M is *1-parallel* [3]). If $\bar{\nabla} \bar{\nabla}h = 0$, then the third fundamental form is said to be *parallel* [18] (*i.e.* M is *2-parallel* [3]).

The mean curvature vector field H of M_r is defined by

$$H = \frac{1}{n} \sum \langle e_i, e_i \rangle h(e_i, e_i), i = 1, \dots, n. \quad (1.7)$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal frame field of M_r . H is said to be parallel when $DH = 0$ holds.

If the second fundamental form h satisfies

$$g(X, Y)H = h(X, Y), \quad (1.8)$$

for any tangent vector fields X, Y of M_r , then M_r is called a totally umbilical. A totally umbilical submanifold with parallel mean curvature vector fields is said to be an *extrinsic sphere* [21].

3 Helices in a Pseudo-Riemannian Manifold

Let γ be a regular curve in a pseudo-Riemannian manifold M_r . We denote the tangent vector field $\gamma'(s)$ by the letter X , when $\langle X, X \rangle = +1$ or -1 , γ is called a *unit speed curve*. The curve γ is called a *Frenet curve of osculating order d* (See [9]) if its derivatives $\gamma'(s), \gamma''(s), \dots, \gamma^{(d)}(s)$ are linearly independent and $\gamma'(s), \gamma''(s), \dots, \gamma^{(d+1)}(s)$ are no longer linearly independent for all $s \in I$.

To each Frenet curve of order d we can associate an orthonormal d frame $\{V_1, V_2, \dots, V_d\}$ along γ , called the *Frenet frame*, and k_1, k_2, \dots, k_{d-1} are *curvature functions* of γ .

Proposition 1 [19]. *If $\gamma : I \longrightarrow M_r$ is a non-null differentiable curve of an n -dimensional pseudo-Riemannian manifold M_r of osculating order d ($0 \leq d \leq n$) and $\{V_1 = X, V_2, \dots, V_d\}$ is the Frenet frame of γ then*

$$V_1' = \nabla_X X = \varepsilon_2 k_1 V_2, \quad (2.1)$$

$$V_2' = \nabla_X V_2 = -\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3, \quad (2.2)$$

$$\vdots$$

$$V_{d-1}' = \nabla_X V_{d-1} = -\varepsilon_{(d-2)} k_{(d-2)} V_{(d-2)} + \varepsilon_d k_{(d-1)} V_d, \quad (2.3)$$

$$V_d' = \nabla_X V_d = -\varepsilon_{(d-1)} k_{(d-1)} V_{(d-1)}, \quad (2.4)$$

where $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$, k_i , $1 \leq i \leq (d-1)$ are curvature functions of γ .

Definition 2 *Let γ be a smooth curve of osculating order d on M_r . The curve γ is called a *W-curve* (or a *helix*) of rank d if k_1, k_2, \dots, k_{d-1} are constant and $k_d = 0$. In particular, a W-curve of rank 2 is called a *geodesics circle*. A W-curve of rank 3 is a *right circular helix* [9].*

Proposition 3 *Let γ be a non-null W-curve in M_r . If γ is of rank 2 then γ''' is a scalar multiple of γ' . In this case necessarily*

$$\gamma'''(s) = -\varepsilon_1 \varepsilon_2 k_1^2 \gamma'(s). \quad (2.5)$$

Proof. By the use of (2.1) we have $\gamma''(s) = \varepsilon_2 k_1 V_2(s)$. Furthermore, differentiating this equation with respect to s and using (2.2) we obtain

$$\gamma'''(s) = -\varepsilon_1 \varepsilon_2 k_1^2 X + \varepsilon_2 k_1' V_2(s) + \varepsilon_2 \varepsilon_3 k_1 k_2 V_3(s). \quad (2.6)$$

Since γ is a W-curve of rank 2 then by definition k_1 is constant and $k_2 = 0$ we get the result. ■

Proposition 4 *Let γ be a non-null W-curve of M_r . If γ is of osculating order 3 then*

$$\gamma''''(s) = -\varepsilon_2 (\varepsilon_1 k_1^2 + \varepsilon_3 k_2^2) \gamma''(s). \quad (2.7)$$

Proof. Differentiating (2.6) and using the fact that k_1, k_2 are constant and $k_3 = 0$ we get the result. ■

4 Planar Geodesic Immersions

Let $f : M_r \rightarrow \mathbb{N}_s$ be an isometric immersion from an n -dimensional, connected pseudo-Riemannian manifold M_r of index r ($0 \leq r \leq n$) into m -dimensional pseudo-Riemannian manifold \mathbb{N}_s of index s . For a point $p \in M_r$ and a unit vector $X \in T_p(M_r)$ the vector X and the normal space $T_p^\perp(M_r)$ determine a $(m - n + 1)$ -dimensional subspace $E(p, X)$ of $T_{f(p)}(\mathbb{N}_s)$ which determines a $(m - n + 1)$ -dimensional totally geodesic submanifold W of \mathbb{N}_s . The intersection of M_r with W gives rise a curve γ (in a neighborhood of p) called the *normal section* of M_r at point p in the direction of X [7].

The submanifold M_r (or the isometric immersion f) is said to have *d-planar normal sections* if for each normal section γ the first, second and higher order derivatives $\gamma'(s), \gamma''(s), \dots, \gamma^{(d)}(s), \gamma^{(d+1)}(s)$; ($1 \leq d \leq m - n + 1$) are linearly dependent as vectors in W [7].

The submanifold M_r is said to have *d-planar geodesic normal sections* if each normal section of M_r is a geodesic of M_r .

In [5] the immersion in pseudo-Euclidean space with 2-planar geodesic normal section have been studied by Blomstrom (See also [10]).

We have the following result.

Proposition 5 *Let γ be a non-null geodesic normal section of M_r . If $\gamma'(s) = X(s)$, then we have*

$$\gamma''(s) = h(X, X), \quad (3.1)$$

$$\gamma'''(s) = -A_{h(X, X)}X + (\bar{\nabla}_X h)(X, X), \quad (3.2)$$

$$\begin{aligned} \gamma''''(s) = & -\nabla_X(A_{h(X, X)}X) - h(A_{h(X, X)}X, X) \\ & - A_{(\bar{\nabla}_X h)(X, X)}X + (\bar{\nabla}_X \bar{\nabla}_X h)(X, X). \end{aligned} \quad (3.3)$$

Example 6 [5] *Pseudo-Riemannian sphere*

$$S_r^n(c) = \left\{ p \in \mathbb{E}_r^{n+1} : \langle p - a, p - a \rangle = \frac{1}{c} \right\}, c > 0, \quad (3.4)$$

and pseudo-Riemannian hyperbolic space

$$H_r^n(c) = \left\{ p \in \mathbb{E}_{r+1}^{n+1} : \langle p - a, p - a \rangle = \frac{1}{c} \right\}, c < 0, \quad (3.5)$$

both have 2-planar geodesic normal sections.

Definition 7 *The submanifold M_r (or the isometric immersion f) is said to be pseudo-isotropic at p if*

$$L = \langle h(X, X), h(X, X) \rangle,$$

is independent of the choice of unit vector X tangent to M_r at p . In particular if L is independent of the points then M_r is said to be constant pseudo-isotropic.

The submanifold M_r is pseudo-isotropic if and only if

$$\langle h(X, X), h(X, Y) \rangle = 0,$$

for any orthonormal vectors X and Y [5].

The following results are well-known.

Theorem 8 [5]. *If the immersion $f : M_r^2 \rightarrow \mathbb{E}_s^m$ has 2-planar geodesic normal sections, then $f(M)$ is a submanifold with zero mean curvature in a hypersphere S_{s-1}^{m-1} or H_{s-1}^{m-1} if and only if L is a non-zero constant.*

Theorem 9 [13]. *The immersion $f : M_r^2 \rightarrow \mathbb{E}_s^m$ with 2-planar geodesic normal sections is constant pseudo-isotropic.*

Theorem 10 [14]. *Let M_r be a pseudo-Riemannian submanifold of index r of a pseudo-Euclidean space \mathbb{E}_s^m of index s with geodesic normal sections. Then*

$$\langle (\bar{\nabla}_X h)(X, X), (\bar{\nabla}_X h)(X, X) \rangle, \quad (3.6)$$

is constant on the their tangent bundle UM of M_r .

Theorem 11 [14]. *Let M_r be a minimal surface of \mathbb{E}_s^5 with geodesics normal sections. Then we have*

- i) M_r is 1-parallel and 0-pseudo isotropic (i.e. $L = 0$),*
- ii) M_r has 2-planar geodesic normal sections,*
- iii) M_r is flat.*

Submanifolds M in \mathbb{R}^{n+d} with 3-planar normal sections have been studied by S.J.Li for the case M is isotropic [15] and sphered [16]. See also [4] for the case M is a product manifold in \mathbb{R}^{n+d} . In [1] the authors consider submanifolds in a real space form $\mathbb{N}^{n+d}(c)$ with 3-planar geodesic normal sections.

We proved the following results.

Lemma 12 *Let $f : M_r \rightarrow \mathbb{N}_s$ be an isometric immersion with 3-planar geodesic normal sections then f is constant pseudo-isotropic.*

Proof. Similar to the proof of Lemma 4.1 in [20]. ■

Proposition 13 *Let $f : M_r \rightarrow \mathbb{N}_s$ be an isometric immersion with 3-planar geodesic normal sections then we have*

$$(\bar{\nabla}_X h)(X, X) = \varepsilon_2(Xk_1)V_2 + \varepsilon_2\varepsilon_3k_1k_2V_3, \quad (3.7)$$

$$A_{h(X,X)}X = \varepsilon_1\varepsilon_2k_1^2X. \quad (3.8)$$

Proof. Let γ be a normal section of M_r at point $p = \gamma(s)$ in the direction of X . We suppose that $k_1(s)$ is positive. Then k_1 is also smooth and there exists a unit vector field V_2 along γ normal to M_r such that

$$h(X, X) = \langle V_2, V_2 \rangle k_1 V_2. \quad (3.9)$$

Since $\bar{\nabla}_X V_2$ is also tangent to M_r , there exists a vector field V_3 normal to M_r and mutually orthogonal to X and V_2 such that

$$\tilde{\nabla}_X V_2 = -\langle X, X \rangle k_1 X + \langle V_3, V_3 \rangle k_2 V_3. \quad (3.10)$$

Differentiating (3.9) covariantly and using (3.10) we get

$$(\bar{\nabla}_X h)(X, X) = -\varepsilon_1 \varepsilon_2 k_1^2 X + \varepsilon_2 (X k_1) V_2 + \varepsilon_2 \varepsilon_3 k_1 k_2 V_3, \quad (3.11)$$

where $\langle V_i, V_i \rangle = \varepsilon_i = \pm 1$. Comparing (3.11) with (3.2) we get the result.

Proposition 14 *Let γ be a normal section of M_r at point $p = \gamma(s)$ in the direction of X . γ is a non-null W-curve of rank 2 in M_r if and only if*

$$\nabla_X \nabla_X X + g(\nabla_X X, \nabla_X X) g(X, X) X = 0. \quad (3.12)$$

Proof. Since $\gamma'(s) = X(s)$, $\gamma''(s) = \nabla_X \nabla_X X$ and

$$g(X, X) = \varepsilon_1, g(\nabla_X X, \nabla_X X) = \varepsilon_2 k_1^2.$$

So, by the use of the equality $\gamma''(s) = \varepsilon_2 k_1 V_2(s)$ we get the result. ■

Proposition 15 *Let M_r be a totally umbilical submanifold of \mathbb{N}_s with parallel mean curvature vector field. If the normal section γ is a W-curve of osculating order 2. Then γ is also a W-curve of \mathbb{N}_s with the same order.*

Proof. Suppose γ is a W-curve of rank 2 in M_r then it satisfies the equality (3.12). Further, by the use of (1.1) we get

$$\gamma'' = \tilde{\nabla}_X X = \nabla_X X + h(X, X). \quad (3.13)$$

Since M_r is totally umbilical then $g(X, X)H = h(X, X)$. So, the equation (3.13) reduces to

$$\gamma'' = \tilde{\nabla}_X X = \nabla_X X + g(X, X)H. \quad (3.14)$$

Differentiating the equation (3.14) with respect to X we obtain

$$\begin{aligned} \gamma''' &= \tilde{\nabla}_X \tilde{\nabla}_X X = \nabla_X \nabla_X X + g(X, \nabla_X X)H \\ &\quad + g(X, X)(-A_H X + D_X H). \end{aligned} \quad (3.15)$$

Further, taking use of $DH = 0$ and (3.13)-(3.15) get

$$\begin{aligned} &\tilde{\nabla}_X \tilde{\nabla}_X X + g(\tilde{\nabla}_X X, \tilde{\nabla}_X X)g(X, X)X \\ &= \nabla_X \nabla_X X - g(H, H)g(X, X)X + \{g(\nabla_X X, \nabla_X X)g(X, X)\}g(X, X)X \\ &= \nabla_X \nabla_X X + g(\nabla_X X, \nabla_X X)g(X, X)X. \end{aligned}$$

So, by previous proposition γ is a W-curve of rank 2 in \mathbb{N}_s . ■

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